SYSTEMS OF EQUATIONS

1. Since both equations contain x^2 and y^2 only, we use elimination. First we multiply both sides of the first equation by (-4) so that when we add equations the x^2 terns will drop out:

$$\left\{\begin{array}{lrcr}(E1) & x^2+y^2 & = & 4\\ (E2) & 4x^2+9y^2 & = & 36\end{array}\right. \xrightarrow[-4E1]{\text{Replace } E1 \text{ with}} \left\{\begin{array}{lrcr}(E1) & -4x^2-4y^2 & = & -16\\ (E2) & 4x^2+9y^2 & = & 36\end{array}\right.$$

$$\left\{\begin{array}{lrcr}(E1) & -4x^2-4y^2 & = & -16\\ (E2) & 4x^2+9y^2 & = & 36\end{array}\right. \xrightarrow{\text{Add } E1 \text{ and} E2} \quad 5y^2 = 20$$

From $5y^2 = 20$, we get $y^2 = 4$ or $y = \pm 2$. To find the associated x values, we substitute each value of y into one of the equations to find the resulting value of x.

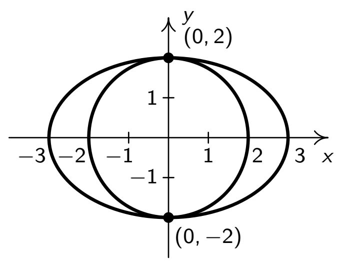
Choosing $x^2 + y^2 = 4$, we find that for both $y = -2$ and $y = 2$, we get $x^2 + (\pm 2)^2 = 4$ or $x^2 + 4 = 4$ so $x^2 = 0$ or $x = 0$. Hence, our solution is $\{(0, 2), (0, -2)\}$.

To verify these answers algebraically, we substitute $(x, y) = (0, 2)$ and $(x, y) = (0, -2)$ into each of the original equations, and they both check.

To check our answer graphically, we sketch both equations and look for their points of intersection.

The graph of $x^2 + y^2 = 4$ is a circle centered at $(0, 0)$ with a radius of 2. To graph $4x^2 + 9y^2 = 36$, we convert to standard form $\frac{x^2}{9} + \frac{y^2}{4} = 1$ and recognize it as an ellipse centered at $(0, 0)$ with a major axis along the x-axis of length 6 and a minor axis along the y-axis of length 4.

We see from the graph that the two curves intersect at their y-intercepts only, $(0, \pm 2)$.



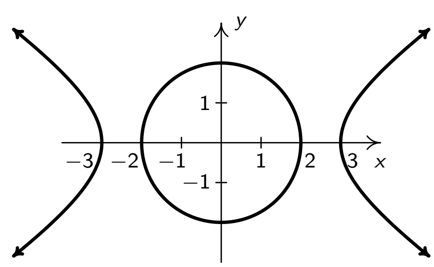
2. Once again, both equations contain x^2 and y^2 only, we use elimination.

$$\left\{\begin{array}{lrcr}(E1) & x^2+y^2 & = & 4\\ (E2) & 4x^2-9y^2 & = & 36\end{array}\right. \xrightarrow[-4E1]{\text{Replace } E1 \text{ with}} \left\{\begin{array}{lrcr}(E1) & -4x^2-4y^2 & = & -16\\ (E2) & 4x^2-9y^2 & = & 36\end{array}\right.$$

$$\left\{\begin{array}{lrcr}(E1) & -4x^2-4y^2 & = & -16\\ (E2) & 4x^2-9y^2 & = & 36\end{array}\right. \xrightarrow{\text{Add } E1 \text{ and} E2} \quad -13y^2 = 20$$

Since the equation $-13y^2 = 20$ has no real solutions, the system has no solution. To verify this graphically, we note that $x^2 + y^2 = 4$ is the same circle as before, but when writing the second equation in standard form, $\frac{x^2}{9} - \frac{y^2}{4} = 1$, we find a hyperbola centered at $(0, 0)$ opening to the left and right with a transverse axis of length 6 and a conjugate axis of length 4.

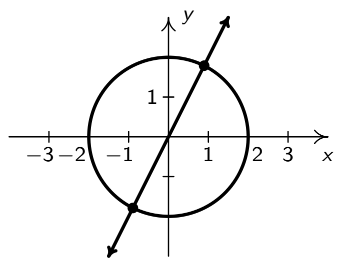
We see that the circle and the hyperbola have no points in common, hence, there are no solutions.



3. Since there are no like terms among the two equations, elimination won't work here. Instead, we proceed using substitution. Solving $y - 2x = 0$ for y, we get $y = 2x$. Substituting this into $x^2 + y^2 = 4$ gives $x^2 + (2x)^2 = 4$. Simplifying, we get $5x^2 = 4$ or $x = \pm\frac{2\sqrt{5}}{5}$.

   Returning to the equation we used for the substitution, $y = 2x$, we find when $x = \frac{2\sqrt{5}}{5}$, $y = \frac{4\sqrt{5}}{5}$, so one solution is $\left(\frac{2\sqrt{5}}{5}, \frac{4\sqrt{5}}{5}\right)$. Likewise, we find when $x = -\frac{2\sqrt{5}}{5}$, $y = -\frac{4\sqrt{5}}{5}$ so the other solution is $\left(-\frac{2\sqrt{5}}{5}, -\frac{4\sqrt{5}}{5}\right)$. Hence, our final answer is $\left\{\left(\frac{2\sqrt{5}}{5}, \frac{4\sqrt{5}}{5}\right), \left(-\frac{2\sqrt{5}}{5}, -\frac{4\sqrt{5}}{5}\right)\right\}$. We can check these solutions algebraically by substituting both pairs into both equations and making sure the equations check.

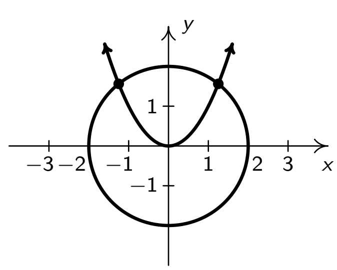
   The graph of $x^2 + y^2 = 4$ is our circle from before and the graph of $y - 2x = 0$, or $y = 2x$ is a line through the origin with slope 2. Even though we cannot easily verify the numerical values of the points of intersection from our sketch, we can be sure there are just two solutions: one in Quadrant I and one in Quadrant III. Using desmos, we can get decimal approximations to the intersection points pretty easily and we can compare those to our exact solutions.



4. While it may be tempting to solve $y - x^2 = 0$ as $y = x^2$ and substitute, we note that this system is set up for elimination. (That being said, it may be worth your time to use substitution since it would review some important skills from Chapter 2.)

$$\left\{ \begin{array}{rrcl} (E1) & x^2 + y^2 & = & 4 \\ (E2) & y - x^2 & = & 0 \end{array} \right. \xrightarrow{\text{Add } E1 \text{ and } E2} \left\{ \begin{array}{rcl} y^2 + y & = & 4 \end{array} \right.$$

   From $y^2 + y = 4$ we get $y^2 + y - 4 = 0$. Using the quadratic formula, we get $y = \frac{-1\pm\sqrt{17}}{2}$. Due to the complicated nature of these answers, it is worth our time to make a quick sketch of both equations first to head off any extraneous solutions we may encounter.



   We see that the circle $x^2 + y^2 = 4$ intersects the parabola $y = x^2$ exactly *twice*, and both of these points have a positive y value. Of the two solutions for y, only $y = \frac{-1+\sqrt{17}}{2}$ is positive, so to get our solution, we substitute this into $y - x^2 = 0$ and solve for x. We get $x = \pm\sqrt{\frac{-1+\sqrt{17}}{2}} = \pm\frac{\sqrt{-2+2\sqrt{17}}}{2}$.

   Our final answer is $\left\{\left(\frac{\sqrt{-2+2\sqrt{17}}}{2}, \frac{-1+\sqrt{17}}{2}\right), \left(-\frac{\sqrt{-2+2\sqrt{17}}}{2}, \frac{-1+\sqrt{17}}{2}\right)\right\}$. Checking these answers algebraically amounts to a true test of anyone's algebraic mettle so checking decimal approximations of intersections using desmos may be a saner approach.